

Problems in algebra inspired by universal algebraic geometry

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Abstract

Let Θ be a variety of algebras. In every Θ and every algebra H from Θ one can consider algebraic geometry in Θ over H . We consider also a special categorical invariant $K_{\Theta}(H)$ of this geometry. The classical algebraic geometry deals with the variety $\Theta = \text{Com} - P$ of all associative and commutative algebras over the ground field of constants P . An algebra H in this setting is an extension of the ground field P . Geometry in groups is related to varieties Grp and $\text{Grp} - G$, where G is a group of constants. The case $\text{Grp} - F$ where F is a free group, is related to Tarski's problems devoted to logic of a free group.

The described general insight on algebraic geometry in different varieties of algebras inspires some new problems in algebra and algebraic geometry. The problems of such kind determine, to a great extent, the content of universal algebraic geometry.

For example, a general and natural problem is:

When do the algebras H_1 and H_2 have the same geometry?

or more specifically,

What are the conditions on algebras from a given variety Θ which provide coincidence of their algebraic geometries?

We consider two variants of coincidence:

- 1) $K_{\Theta}(H_1)$ and $K_{\Theta}(H_2)$ are isomorphic.
- 2) These categories are equivalent.

This problem is highly connected with the following general algebraic problem:

Let Θ^0 be the category of all free in Θ algebras $W = W(X)$, where X is finite. Consider the groups of automorphisms $\text{Aut}(\Theta^0)$

for different varieties Θ and also the groups of autoequivalences of Θ^0 . The problem is to describe these groups for different Θ .

We start with the short overview of main definitions and results and then consider the list of unsolved problems. The results without references can be found in [23]

1 Definitions

1.1. Fix a variety Θ . Take an algebra $H \in \Theta$ and a free in Θ algebra $W = W(X)$ with finite X . The set of homomorphisms $\text{Hom}(W, H)$ we consider as an affine space of points over H . Points of this space are the homomorphisms $\mu : W \rightarrow H$. If $X = \{x_1, \dots, x_n\}$, then we have a bijection

$$\alpha_X : \text{Hom}(W, H) \rightarrow H^{(n)},$$

defined by $\alpha_X(\mu) = (\mu(x_1), \dots, \mu(x_n))$. A point μ is a root of the pair (w, w') , $w, w' \in W$, if $w^\mu = w'^\mu$, which means also that $(w, w') \in \text{Ker}\mu$. Here $\text{Ker}\mu$ is, in general, a congruence of the algebra W . Simultaneously, μ is a solution of the equation $w = w'$. We will identify the pair (w, w') and the equation $w = w'$.

Let now T be a system of equations in W and A a set of points in $\text{Hom}(W, H)$. We have the following Galois correspondence

$$\begin{cases} T'_H = \{\mu : W \rightarrow H \mid T \subset \text{Ker}\mu\} \\ A'_W = \bigcap_{\mu \in A} \text{Ker}\mu \end{cases}$$

Definition 1.1. The set A of the form $A = T'$ for some T we call a (closed) algebraic set. The congruence T of the form $T = A'$ for some A is an H -closed congruence.

It is easy to see that the congruence T is H -closed if and only if $W/T \in SC(H)$, where S and C are the operators of taking subgroups and cartesian products on group classes.

One can consider the closures $A'' = (A')'$ and $T''_H = (T'_H)'$.

Proposition 1.1. *The pair (w_0, w'_0) belongs to T''_H if and only if the formula (infinitary quasiidentity)*

$$\left(\bigwedge_{((w, w') \in T} (w \equiv w') \right) \Rightarrow w_0 \equiv w'_0$$

holds in H .

1.2. We have defined the category Θ^0 . Let us add to the definition that for all objects of Θ^0 the finite X are subsets of an infinite universum X^0 . Then Θ^0 is a small category.

Define, further, the *category of affine spaces* $K_\Theta^0(H)$. Objects of this category are affine spaces

$$\text{Hom}(W, H), \quad W \in \text{Ob } \Theta^0.$$

The morphisms

$$\tilde{s} : \text{Hom}(W(X), H) \rightarrow \text{Hom}(W(Y), H)$$

of $K_\Theta^0(H)$ are determined by homomorphisms $s : W(Y) \rightarrow W(X)$ by the rule $\tilde{s}(\nu) = \nu s$ for every $\nu : W(X) \rightarrow H$. We have a contravariant functor

$$\varphi : \Theta^0 \rightarrow K_\Theta^0(H).$$

Proposition 1.2. *The functor $\varphi : \Theta^0 \rightarrow K_\Theta^0(H)$ determines duality of categories if and only if $\text{Var}(H) = \Theta$.*

Proceed now to the *category of algebraic sets* $K_\Theta(H)$. Its objects have the form (X, A) , where A is an algebraic set in the space $\text{Hom}(W(X), H)$. The morphisms $[s] : (X, A) \rightarrow (Y, B)$ are defined by those $s : W(Y) \rightarrow W(X)$, for which $\tilde{s}(\nu) \in B$ if $\nu \in A$. Simultaneously, we have mappings $[s] : A \rightarrow B$.

Let us define the category $C_\Theta(H)$. Its objects have the form W/T , where $W \in \text{Ob } \Theta^0$ and T is an H -closed congruence in W . Morphisms of $C_\Theta(H)$ are the homomorphisms of algebras.

It is proved that if $\text{Var}(H) = \Theta$ then the transitions $(X, A) \rightarrow W(X)/A'$ and $W/T \rightarrow (X, T'_H)$ determine duality of the categories $K_\Theta(H)$ and $C_\Theta(H)$. In this case the category Θ^0 is a subcategory in $C_\Theta(H)$. The skeleton of the category $K_\Theta(H)$ is denoted by $\tilde{K}_\Theta(H)$. This category is the category of *algebraic varieties* over H . Correspondingly, the category $\tilde{C}_\Theta(H)$ is defined.

The category $K_\Theta^0(H)$ is always a subcategory in $K_\Theta(H)$ [23].

We consider also the categories K_Θ and C_Θ where the algebra H is not fixed. Correspondingly, we have the categories \tilde{K}_Θ and \tilde{C}_Θ .

1.3 Consider a functor $Cl_H : \Theta^0 \rightarrow \mathbf{poSet}$, where \mathbf{poSet} denotes the category of partially ordered sets.

This functor corresponds to every algebra H in Θ . By definition, for every $W \in \text{Ob } \Theta^0$ the poset $Cl_H(W)$ is the set of all H -closed congruences T in W with the natural ordering. Correspondingly, there is a lattice $Cl_H(W)$.

Let now a morphism

$$s : W(Y) \rightarrow W(X)$$

be given in Θ^0 . It corresponds a map

$$Cl_H(s) : Cl_H(W(X)) \rightarrow Cl_H(W(Y)),$$

defined by the rule $Cl_H(s)(T) = s^{-1}T$. Here $T \in Cl_H(W(X))$; $s^{-1}T$ is a congruence in $W(Y)$, defined by the rule $w(s^{-1}T)w'$ if and only if $w^sTw'^s$, $w, w' \in W(Y)$. The congruence $s^{-1}T$ is also H -closed and the mapping $Cl_H(s)$ is a morphism in the category **poSet**.

This defines a contravariant functor Cl_H , which plays an important part in the sequel.

In the same way one can consider a covariant functor $Als_H : \Theta^0 \rightarrow \mathbf{poSet}$, where $Als_H(W)$ is the poset of algebraic sets in the affine space $\text{Hom}(W, H)$.

2 General look at the theory

The main concepts of the theory are as follows:

1. *Geometric properties of algebras H in Θ . An algebra H is considered in respect to its geometry and equations over H .*
2. *Geometric relations between algebras in Θ .*
3. *Structure of algebraic sets for every given algebra H and every W . The lattice of algebraic sets in the given affine space.*

We will focus our attention on the problems related to Items 1 and 2. The item 3 is a separate topic which requires the additional clarity.

We now quote some working notions around which the theory rotates.

First of all these are geometrical invariants of an algebra H : *special categories and functors*. Categories are presented by the *categories of algebraic sets and algebraic varieties* $K_\Theta(H)$ and $\tilde{K}_\Theta(H)$. They are related to the *categories* $C_\Theta(H)$ and $\tilde{C}_\Theta(H)$. Another invariant of algebras is a contravariant functor $Cl_H : \text{Var}(H)^0 \rightarrow \mathbf{poSet}$. Categories K_Θ and C_Θ are invariants of the whole variety Θ .

The main features of algebras H_i we are dealing with are *geometrical noetherianity*, *logical noetherianity* and *geometrical distributivity*. Relations between algebras are presented by the notions of *geometrical equivalence*, *geometrical similarity*, *geometrical compatibility*, *coincidence of geometries* and *coincidence of lattices*. Here, coincidence of lattices in the most general case is defined as isomorphism of functors of the $Cl_{H_1} \rightarrow Cl_{H_2}\varphi$ type, where φ is an isomorphism of categories $\varphi : Var(H_1)^0 \rightarrow Var(H_2)^0$.

Further we give all necessary definitions.

2.1 Examples of geometrical properties and relations

2.1. Geometrical equivalence.

Definition 2.1. Algebras H_1 and H_2 from Θ are called *geometrically equivalent* if for every $W = W(X) \in Ob \Theta^0$ and every T in W , we have

$$T''_{H_1} = T''_{H_2}.$$

This means also that $Cl_{H_1} = Cl_{H_2}$.

It is clear that if the algebras H_1 and H_2 are geometrically equivalent, then the categories $C_\Theta(H_1)$ and $C_\Theta(H_2)$ coincide. Correspondingly, the categories $K_\Theta(H_1)$ and $K_\Theta(H_2)$ are isomorphic.

Theorem 1. *Algebras H_1 and H_2 are geometrically equivalent if and only if*

$$LSC(H_1) = LSC(H_2).$$

Here the operator L on classes of algebras is defined in the usual local sense, i.e., for every class \mathfrak{X} an algebra G belongs to \mathfrak{X} , if every finitely generated subalgebra H of G belongs to \mathfrak{X} . It can be proved that $LSC(\mathfrak{X}) = \tilde{q}Var(\mathfrak{X})$, where $\tilde{q}Var(\mathfrak{X})$ is the class of algebras which is determined by infinitary quasiidentities of the class \mathfrak{X} . Correspondingly, $qVar(\mathfrak{X})$ is the quasivariety which is generated by the class \mathfrak{X} . Hence, geometrical equivalence of algebras means also that

$$\tilde{q}Var(H_1) = \tilde{q}Var(H_2),$$

i.e., H_1 and H_2 have the same infinitary quasiidentities.

2.2. Geometrically and logically noetherian algebras.

Definition 2.2. An algebra $H \in \Theta$ is called *geometrically noetherian* if for an arbitrary W and T in W there exists a finite $T_0 \subset T$ such that

$$T_H'' = (T_0)_H''.$$

An algebra H is geometrically noetherian if and only if for every W and T in W there exists a finite subset $T_0 \subset T$ such that

$$\left(\bigwedge_{((w,w') \in T)} (w \equiv w') \right) \Rightarrow w_0 \equiv w'_0$$

holds in H if and only if the quasiidentity

$$\left(\bigwedge_{((w,w') \in T_0)} (w \equiv w') \right) \Rightarrow w_0 \equiv w'_0$$

holds in H . Here T_0 is independent from (w_0, w'_0) .

Definition 2.3. In case when T_0 depends on (w_0, w'_0) we call H *logically noetherian*.

The notion to be logically noetherian means also that T'' coincides with $\bigcup T_\alpha''$ where the union is taken over all finite subsets T_α in T .

Obviously, if H is geometrically noetherian, then H is logically noetherian.

An algebra H turns to be *geometrically noetherian* if and only if in every $W = W(X)$ the ascending chain condition for H -closed congruences holds. Dually, the descending chain condition for algebraic sets in $\text{Hom}(W(X), H)$ holds in geometrically noetherian algebras. An algebra H is logically noetherian if the union of a directed set of H -closed congruences is also H -closed.

Theorem 2. [19] Let H_1 and H_2 be logically noetherian algebras. They are geometrically equivalent if and only if $q\text{Var}(H_1) = q\text{Var}(H_2)$.

Theorem 3. [19] If $H \in \Theta$ is not logically noetherian, then there exists an ultrapower H' of H such that the algebras H and H' are not geometrically equivalent.

However, these algebras have the same elementary theories and, in particular, the same quasiidentities.

These theorems lead to the following general problem:

*For which varieties Θ there exist non-logically noetherian algebras in Θ ?
How often these algebras can appear?*

The existence of such phenomenon for groups is proved in the paper by K.Gobel, S.Shelah [7]. The idea of their proof is based on the existence of the continuum different 2-generated simple groups [9]. For representations of groups the result is proved by A.Tsurkov [25]. For associative algebras over a field the result also holds [24]. In the recent paper of Lichtman - Passman [10] the existence of the continuum of 3-generated simple algebras is proved.

The results and notions above are of universal character. In particular, they can be applied to multi-sorted algebras. Further we consider concrete Θ and mostly for them we formulate problems.

3 Geometrical properties of algebras. Problems

Problem 1. *Let $G = AwrB$ be a wreath product of some groups A and B .*

1. *When G is geometrically noetherian?*
2. *When G is logically noetherian but not geometrically noetherian?*
3. *Are there groups $G = AwrB$ which are not logically noetherian for some appropriate A and B .*

It is known that any free group $W(X)$ is geometrically noetherian (Guba [8]). Moreover, every group or algebra which admits faithful finite dimensional representation is geometrically noetherian (Miasnikov-Remeslennikov [19], Kanel-Belov). Every finite dimensional representation of a group is geometrically noetherian (Tsurkov [25],

Problem 2.

Is it true that every free Lie algebra $W(X)$ is geometrically noetherian?

Most likely the answer is negative. Thus arises the following:

Problem 3.

Is it true that every free Lie algebra $W(X)$ is logically noetherian?

Problem 4.

Is it true that every free associative algebra $W(X)$ is geometrically noetherian?

Here the expected answer is also seems to be negative. Then:

Problem 5.

Is it true that every free associative algebra $W(X)$ is logically noetherian?

Any two free groups have the same quasi-identities. The similar fact is valid for free associative and free Lie algebras. Free groups are also geometrically noetherian. Geometrical noetherianity together with coincidence of quasi-identities implies that any two free groups are geometrically equivalent. Thus, the free groups have the same logic of quasi-identities and the same geometry. The positive solution of Problems 3 and 5 would mean that the same fact holds true for the free Lie algebras and free associative algebras.

Problem 6.

Is it true that there exists a continuum of different k -generated simple Lie algebras? Here k is fixed.

Problem 7.

Is it true that there exists a non-logically noetherian Lie algebra?

The next problems are devoted to lattices of algebraic sets.

Definition 3.1. An algebra H is called *geometrically distributive* if for every W the lattice of algebraic sets $Als_H(W)$ (and the lattice $Cl_H(W)$, respectively) is distributive.

The *geometrically modular* algebras are defined in the similar way.

Problem 8.

Which algebras H are geometrically distributive?

This problem makes sense for groups, groups with the fixed group of constants, and for other varieties Θ .

Problem 9.

Which algebras H are geometrically modular?

We introduced earlier the category K_Θ of algebraic sets without the fixed set H .

Problem 10.

When the categories K_{Θ_1} and K_{Θ_2} are isomorphic and when they are equivalent? Consider separately the case when Θ_1 and Θ_2 are subvarieties of a bigger variety Θ .

This problem should be related with the known results of McKenzie [18] about equivalence of two varieties of algebras.

4 Other geometrical relations between algebras

We have defined the notion of geometric equivalence of algebras H_1 and H_2 . Now we define two more general notions.

4.1. First we recall the definition of isomorphism of functors.

Let two functors $\varphi_1, \varphi_2 : C_1 \rightarrow C_2$ of the categories C_1, C_2 be given. A *homomorphism (natural transformation) of functors* $s : \varphi_1 \rightarrow \varphi_2$ is a function, relating a morphism in C_2 , denoted by $s_A : \varphi_1(A) \rightarrow \varphi_2(A)$ to every object A of the category C_1 . For every $\nu : A \rightarrow B$ in C_1 there is a commutative diagram

$$\begin{array}{ccc} \varphi_1(A) & \xrightarrow{s_A} & \varphi_2(A) \\ \varphi_1(\nu) \downarrow & & \downarrow \varphi_2(\nu) \\ \varphi_1(B) & \xrightarrow{s_B} & \varphi_2(B) \end{array}$$

in the case of covariant φ_1 and φ_2 . For contravariant φ_1 and φ_2 the corresponding diagram is

$$\begin{array}{ccc} \varphi_1(B) & \xrightarrow{s_B} & \varphi_2(B) \\ \varphi_1(\nu) \downarrow & & \downarrow \varphi_2(\nu) \\ \varphi_1(A) & \xrightarrow{s_A} & \varphi_2(A) \end{array}$$

An invertible homomorphism $s : \varphi_1 \rightarrow \varphi_2$ is called *an isomorphism (natural isomorphism) of functors*. The isomorphism property holds if $s_A : \varphi_1(A) \rightarrow \varphi_2(A)$ is an isomorphism in C_2 for any A .

Definition 4.1. Algebras H_1 and H_2 are called *geometrically similar*, if there exists an automorphism $\varphi : \Theta^0 \rightarrow \Theta^0$ such that there is a *correct isomorphism* of functors $\alpha(\varphi) : Cl_{H_1} \rightarrow Cl_{H_2}\varphi$.

Here, correctness means compatibility with the automorphism φ . Namely, let $s_1, s_2 : W_1 \rightarrow W_2$ be given, and T be H_1 -closed in W_2 . Denote, $T^* = \alpha(\varphi)_{W_1}(T)$. There are canonical homomorphisms $\mu_T : W_2 \rightarrow W_2/T$ and $\mu_{T^*} : \varphi(W_2) \rightarrow \varphi(W_2)/T^*$. Correctness means that $\mu_T s_1 = \mu_{T^*} s_2$ holds if and only if $\mu_{T^*} \varphi(s_1) = \mu_{T^*} \varphi(s_2)$.

Definition 4.2. Algebras H_1 and H_2 are called *geometrically compatible*, if there exists an autoequivalence of the category $\Theta^0 \xrightarrow{\psi} \Theta^0$ such that there are the natural transformations of functors

$$\alpha(\varphi) : Cl_{H_1} \rightarrow Cl_{H_2}\varphi,$$

$$\alpha(\psi) : Cl_{H_2} \rightarrow Cl_{H_1}\psi,$$

which are compatible as before with φ and ψ .

4.2. We consider also *the correct isomorphisms* of the categories $C_\Theta(H_1) \rightarrow C_\Theta(H_2)$. These are isomorphisms which induce an automorphism of the category Θ^0 . The correct isomorphisms of the categories $K_\Theta(H_1) \rightarrow K_\Theta(H_2)$ are defined in a similar way.

Theorem 4. Suppose $VarH_1 = VarH_2 = \Theta$. The categories $K_\Theta(H_1)$ and $K_\Theta(H_2)$ are correctly isomorphic if and only if H_1 and H_2 are geometrically similar.

Theorem 5. Suppose $VarH_1 = VarH_2 = \Theta$. The categories $K_\Theta(H_1)$ and $K_\Theta(H_2)$ are correctly equivalent if and only if H_1 and H_2 are geometrically compatible.

Correctness here means also that the lattices of algebraic sets for H_1 and H_2 are the same.

These theorems are of the universal character. Each of them should be specified for particular varieties Θ . This specialization depends very much on the description of the group $\text{Aut}(\Theta^0)$.

Problem 11

Consider the similar problems without assumption of correctness for isomorphisms and equivalences of categories.

5 $\text{Aut}(\Theta^0)$

All automorphisms of the category Θ^0 are known in the following cases (see [3],[14],[15], [13], [11], [24], [17]) .

1. *Groups.*
2. *Groups with a free group of constants, $\text{Grp} - F$.*
3. *Associative and commutative algebras, $\text{Com} - P$.*
4. *Associative algebras.*
5. *Lie algebras.*
6. *K -modules, K is an IBN- ring.*
7. *Semigroups.*

In the situation of Lie algebras the description of automorphisms uses the description of the group $\text{Aut}(\text{End}(W(x, y)))$. This observation motivates the following:

Problem 12

Study the group $\text{Aut}(\text{End}(W(X)))$, where $W(X)$ is the free Lie algebra over a finite set X .

Problem 13

Study the group $\text{Aut}(\Theta^0)$ for various interesting subvarieties of the variety of all groups. For example for the varieties \mathfrak{N}_c , \mathfrak{A}^2 , etc.

Problem 14

Study the group $\text{Aut}(\Theta^0)$ for various interesting subvarieties of the variety of all Lie algebras.

Problem 15

Study the group $\text{Aut}(\Theta^0)$ for various interesting subvarieties of the variety of all associative algebras.

6 Algebras with the same algebraic geometry

Recall that we look at the notion of coincidence of geometries in two variants.

1. The categories $K_{\Theta}(H_1)$ and $K_{\Theta}(H_2)$ are isomorphic.
2. The categories $K_{\Theta}(H_1)$ and $K_{\Theta}(H_2)$ are equivalent.

In fact, the second case means that the categories of algebraic varieties $\tilde{K}_{\Theta}(H_1)$ and $\tilde{K}_{\Theta}(H_2)$ are isomorphic.

We consider the specific varieties Θ . The problem of coincidence of geometries is solved in the following cases:

1. *For the classical algebraic geometry [3].*
2. *For the non-commutative algebraic geometry related to the variety of all associative algebras [15].*
3. *For the algebraic geometry in the variety of all Lie algebras [24].*
4. *For the geometry in the variety of all groups [24].*
5. *For the variety $\text{Grp} - F$ [4].*

Problem 16.

Investigate coincidence of geometries for some subvarieties of the variety of all groups.

Problem 17.

Investigate coincidence of geometries for some subvarieties of the variety of all Lie algebras.

Problem 18.

Investigate coincidence of geometries for some subvarieties of the variety of all associative algebras. For example, for the subvariety Θ given by a single polynomial identity.

Solution of these problems heavily depends on the solution of the problem of the group $\text{Aut}(\Theta^0)$ description.

7 Coincidence of the lattices of algebraic sets

We consider the following variants for the definition of lattices coincidence.

1. Coincidence of the functors Cl_{H_1} and Cl_{H_2} .
2. Isomorphism of the functors Cl_{H_1} and Cl_{H_2} .

3. The functor Cl_{H_1} is isomorphic to $\text{Cl}_{H_2}\varphi$, where φ is an automorphism of the category Θ^0 .

In the first case the algebras H_1 and H_2 are geometrically equivalent and the lattices in W corresponding to H_1 and H_2 coincide.

In the second case an isomorphism of functors $\alpha : \text{Cl}_{H_1} \rightarrow \text{Cl}_{H_2}$ provides an isomorphism of the corresponding lattices for every W . Besides, there is a compatibility with the morphisms.

In the third case for every W there is an isomorphism of the lattices $\text{Cl}_{H_1}(W)$ and $\text{Cl}_{H_2}(\varphi(W))$.

Problem 19 *For which algebras H_1 and H_2 there is an isomorphism of the functors Cl_{H_1} and Cl_{H_2} .*

Problem 20 *For which algebras H_1 and H_2 there is an isomorphism between Cl_{H_1} and $\text{Cl}_{H_2}\varphi$ for some $\varphi : \Theta^0 \rightarrow \Theta^0$.*

If the algebras H_1 and H_2 are geometrically similar then such an isomorphism exists. Thus, if H_1 and H_2 have the same geometries then the corresponding lattices coincide. The converse statement is not true and this makes everything more attractive.

The problems above seem new also for the classical situation $\text{Com} - P$, where L_1 and L_2 are two extensions of a ground field P .

In particular, what can be said about L_1 and L_2 if for every W the lattices $\text{Cl}_{L_1}(W)$ and $\text{Cl}_{L_2}(W)$ are isomorphic?

Coincidence of these lattices means that L_1 and L_2 are geometrically equivalent. Hence, in this case the logics of quasiidentities for L_1 and L_2 are the same. But we are interested in conditions providing isomorphism of lattices.

8 Representations

8.1. Let K be a commutative, associative ring with unit. We consider the category-variety $\Theta = \text{Rep} - K$. General references on the theory in question are [26],[22],[27], [28].

Objects of this category are representations (V, G) , where V is a K -module and G is a group acting on V . These (V, G) are two-sorted algebras.

The action G on V is denoted by \circ and for every $a \in V$ and $g \in G$ we have $a \circ g \in V$. The action \circ satisfies the natural identities.

Morphisms in $\Theta = \text{Rep} - K$ have the form

$$\mu = (\alpha, \beta) : (V_1, G_1) \rightarrow (V_2, G_2)$$

where $\alpha \in \text{Hom}_K(V_1, V_2)$, $\beta \in \text{Hom}(G_1, G_2)$, and $(a \circ g)^\alpha = a^\alpha \circ g^\beta$.

$\text{Ker}\mu = (\text{Ker}\alpha, \text{Ker}\beta) = (V_0, H)$ is a congruence in (V_1, G_1) in the following sense: H_0 is G_1 -invariant submodule in V_1 and H acts trivially in V_1/V_0 . We have the factor-representation $(V_1, G_1)/(V_0, H) = (V_1/V_0, G/H)$ with the natural theorem on homomorphisms. For a given set $\mu_i = (\alpha_i, \beta_i) : (V_1, G_1) \rightarrow (V_2, G_2)$ we have $\bigcap \text{Ker}\mu_i = (\bigcap \text{Ker}\alpha_i, \bigcap \text{Ker}\beta_i)$.

Free objects W in the category Θ are denoted by $W = W(X, Y)$, where X and Y is a pair of sets. Here:

$$W(X, Y) = (XKF(Y), F(Y))$$

where $F = F(Y)$ is the free group over Y , KF is the group algebra, XKF is the free KF -module over the set X .

For every $w \in XKF$, $w = x_1u_1 + \dots + x_nu_n$, $u_i \in KF$, and $f \in F$ we have

$$w \circ f = x_1(u_1f) + \dots + x_n(u_nf).$$

This is the free representation in the categorical sense over the two-sorted set (X, Y) . The two-sorted equality $w \equiv 0$ is considered as an *action-type equality*, while $f \equiv 1$ is a group equality.

In the book [26] the action-type varieties of representations have been considered. These varieties lie in $\text{Rep} - K$ and can be defined by identities of the type $x \circ u$.

8.2. In $\text{Rep} - K$ one can consider the different operations: Cartesian-direct products, free products-coproducts, subrepresentations, quotients, etc. The following two constructions are of the special type.

Triangular products. For the given representations (V_1, G_1) and (V_2, G_2) consider their triangular product $(V_1, G_1) \nabla (V_2, G_2)$. This is the representation $(V_1 + V_2, G)$, where $g \in G$ has the form

$$g = \begin{bmatrix} g_2 & \varphi g_1 \\ 0 & g_1 \end{bmatrix} = \begin{bmatrix} 1 & \varphi \\ 0 & 1 \end{bmatrix} \begin{bmatrix} g_2 & 0 \\ 0 & g_1 \end{bmatrix},$$

$g_1 \in G_1$, $g_2 \in G_2$, $\varphi \in \text{Hom}(V_2, V_1)$.

For $a \in V_1$, $b \in V_2$ we have $a \circ g = a \circ g_1$; $b \circ g = b \circ g_2 + (b\varphi) \circ g_1$. Here, (V_1, G) is related to (V_1, G_1) and $(V_1 + V_2/V_1, G)$ to (V_2, G_2) .

We consider also wreath product $(V, H)wrG = (V^G, HwrG)$.

8.3. In $\text{Rep} - K$ we consider varieties of the general form and action-type varieties. For the last ones the semigroup \mathfrak{M} of such the varieties \mathfrak{X} is treated. The multiplication in \mathfrak{M} is defined by the rule: $(V, G) \in \mathfrak{X}_1 \mathfrak{X}_2$ if for some G -invariant submodule $V_0 \subset V$ we have $(V_0, G) \in \mathfrak{X}_1$, $(V/V_0, G) \in \mathfrak{X}_2$.

If K is a field then the semigroup \mathfrak{M} is a free semigroup and we have here

$$\text{Var}((V_1, G_1) \nabla (V_2, G_2)) = \text{Var}(V_1, G_1) \text{Var}(V_2, G_2).$$

Let now \mathfrak{N} be the semigroup of group varieties. For $\mathfrak{X} \in \mathfrak{M}$ and $\Theta \in \mathfrak{N}$ we consider the product $\mathfrak{X} \times \Theta \in \mathfrak{M}$ defined as follows: $(V, G) \in \mathfrak{X} \times \Theta$ if for some invariant subgroup H in G we have $(V, H) \in \mathfrak{X}$, $G/H \in \Theta$. Now \mathfrak{N} acts in \mathfrak{M} as a semigroup of endomorphisms of \mathfrak{M} .

The principal theorem in this theory says that the action of \mathfrak{N} in \mathfrak{M} is free. This means also that every \mathfrak{X} in \mathfrak{M} can be uniquely presented in the form

$$\mathfrak{X} = (\mathfrak{X}_1 \times \Theta_1) \dots (\mathfrak{X}_n \times \Theta_n),$$

where all \mathfrak{X}_i are irreducible.

The triangular products and wreath products play the crucial role in the proof of the theorem above.

9 Algebraic geometry in representations

9.1. We consider $\text{Hom}(W, (V, G))$ as the *affine space* over the given representation (V, G) . Here, $W = W(X, Y)$ is the free representation over the finite sets X and Y . *Points* here are homomorphisms

$$\mu : W \rightarrow (V, G).$$

Take $T = (T_1, T_2)$, where T_1 is a set of action-type equalities in W and T_2 is a set of group equalities.

Denote

$$\begin{cases} T'_{(V, G)} = A = \{\mu = (\alpha, \beta) : W \rightarrow (V, G) \mid T_1 \subset \text{Ker} \alpha, T_2 \subset \text{Ker} \beta\} \\ A'_W = T = \bigcap_{\mu \in A} \text{Ker} \mu = \left(\bigcap_{\alpha} \text{Ker} \alpha, \bigcap_{\beta} \text{Ker} \beta \right) \end{cases}$$

$$T_1 = \bigcap \text{Ker} \alpha, \quad T_2 = \bigcap \text{Ker} \beta.$$

Let $\text{Id}_G(F)$ be the verbal subgroup of all identities of the group G in $F = F(Y)$. In every case we have $\text{Id}_G(F) \subset T_2$.

A set A of the form $A = T'$ is an *algebraic set*, and $T = A'$ is a (V, G) -closed congruence in W . The definitions above specialize the general definitions from universal algebraic geometry for the case of the variety-category of group representations. Some results of the universal algebraic geometry in multi-sorted Θ are applicable in this case.

9.2. Now we consider action-type AG in representations.

For the given $W = W(X, Y) = (XKF, F)$ $F = F(Y)$ we take a set $T \subset XKF$. We view T as a set of action-type equalities.

Denote

$$\begin{cases} T^v = A = \{\mu = (\alpha, \beta) : W \rightarrow (V, G) \mid T \subset \text{Ker} \alpha\} \\ A^v = T = \bigcap_{\alpha} \text{Ker} \alpha. \end{cases}$$

Here $A = T^v$ is an *action-type algebraic set*, and $T = A^v$ is an *action-type* (V, G) -closed F -invariant submodule in XKF .

It is easy to see that an algebraic set A is an action-type algebraic set if and only if all points of the type $(0, \beta)$ belongs to A . From this follows that if A is an action-type algebraic set, then

$$A' = (A^v, \text{Id}_G(F)).$$

As before one can define the notions of the geometrically equivalent representations as well as the notions of geometrically and logically noetherian representations. These definitions refer to general case and also to action-type case.

We can consider also the categories $K_{\Theta}(V, G)$ and $C_{\Theta}(V, G)$ for the general case and the categories $K_{\Theta}^{at}(V, G)$ and $C_{\Theta}^{at}(V, G)$ for the action-type case. Here, $\Theta = \text{Rep} - K$ or a subvariety of $\Theta = \text{Rep} - K$.

10.3. Once again the open problems.

Problem 21 *When the representations (V_1, G_1) and (V_2, G_2) have the same geometry.*

This question relates to general situation and to action-type case.

With this problem the notions of geometrically similar and geometrically compatible representations are associated. Furthermore, the latter notions are connected with the automorphisms and autoequivalences of the category $\Theta^0 = (\text{Rep} - K)^0$.

So, we have the following

Problem 22

Investigate the group $\text{Aut}(\text{Rep} - K)^0$.

Recall that an automorphism of the category C is called *inner* if it is isomorphic to unity automorphism 1_C .

Inner automorphisms constitute an invariant subgroup in the group $\text{Aut}(\text{Rep} - K)^0$.

One can speak also on semi-inner automorphisms. In their definition the automorphisms σ of the ring K take part. These automorphisms form a subgroup in the group $\text{Aut}(\text{Rep} - K)^0$. There is also a special mirror (anti-inner) automorphism δ . The existence of such automorphism is based on the consideration of the opposite group and opposite representation.

The problem is to prove that the group $\text{Aut}(\text{Rep} - K)^0$ is generated by automorphisms above.

Repeating the arguments from [24] one can prove that if the similarity of two representations (V_1, G_1) and (V_2, G_2) is related to an inner automorphism φ of the category $(\text{Rep} - K)^0$, then the representations are geometrically equivalent in general. This implies that they are also action-type equivalent.

Problem 23

What is the relation between the representations (V_1, G_1) and (V_2, G_2) if they are geometrically similar and the similarity is based on an semi-inner automorphism φ of the category $(\text{Rep} - K)^0$?

This problem is connected with

Problem 24

Investigate the group $\text{Aut}(\text{End}(KF, F))$.

Let us discuss this problem in more detail. For every representation (V, G) we have the group $\text{Aut}(\text{End}(V, G))$. Let $\xi = (s, \tau)$ be an invertible element of the semigroup $\text{End}(V, G)$. Then ξ is also an automorphism of the representation (V, G) . An inner automorphism $\hat{\xi}$ of the semigroup $\text{End}(V, G)$ corresponds to it.

For every $\mu = (\alpha, \beta) \in \text{End}(V, G)$ we have

$$\hat{\xi}(\mu) = \xi\mu\xi^{-1} = (s\alpha s^{-1}, \tau\beta\tau^{-1}).$$

All these $\hat{\xi}$ form a normal subgroup in the group $\text{Aut}(\text{End}(V, G))$.

Consider the pairs $\xi = (s, \tau)$, where s is a semi-automorphism of the K -module V and $\tau \in \text{Aut}(G)$. There is $\sigma \in \text{Aut}(K)$ such that for every $\lambda \in K$ and $a \in V$ we have:

$$s(\lambda a) = \lambda^\tau s(a).$$

Besides, $(a \circ g)^s = a^s \circ g^\tau$ for every a and g . Here, ξ is a semi-automorphism of the representation (V, G) , it does not belong to the semigroup $\text{End}(V, G)$. However, ξ induces automorphism $\hat{\xi}$ of this semigroup. Here, $\hat{\xi}$ is a semi-inner automorphism of the semigroup $\text{End}(V, G)$ and all these automorphisms form a subgroup in $\text{Aut}(V, G)$.

Let, further, φ be an arbitrary automorphism of the semigroup $\text{End}(V, G)$. For every $\mu = (\alpha, \beta) \in \text{End}(V, G)$ we have

$$\varphi(\mu) = (\varphi_1(\mu), \varphi_2(\mu)).$$

Here, $\varphi_1 : \text{End}(V, G) \rightarrow \text{End}V$, $\varphi_2 : \text{End}(V, G) \rightarrow \text{End}V$ are the homomorphisms of semigroups.

One can calculate the conditions on homomorphisms φ_1 and φ_2 which provide the homomorphism φ of the semigroup $\text{End}(V, G)$. However, it is not clear how to deduce from these conditions the "real constructions".

All above can be applied to the representations of the kind (KG, G) and, in particular, to (KF, F) . In this important case one has to take into account the theorem of Formanek [6], which says that all automorphisms of the semigroup $\text{End}(F)$ are inner. Is it possible to state that all automorphisms of the semigroup $\text{End}(KF, F)$ are semi-inner or of the form $\varphi\delta$ where φ is semiinner.? Or one can construct a counter example?

Now consider the problems of the different kind. All these problems should be stated separately for the general and action-type cases.

Problem 25

Consider the representations (V_1, G_1) and (V_2, G_2) from the point of view of coincidence of the corresponding lattices of algebraic varieties.

Problem 26

Is it true that the representation (XKF, F) is geometrically noetherian or logically noetherian?

Recall here that the group F is geometrically noetherian.

Problem 27

Consider the notions of geometrical and logical noetherianity in respect to triangular products of representations and wreath products of a representation and a group.

Problem 28

Let G be not logically noetherian group. Whether the group algebra PG is also not logically noetherian for some field P . The same question is meaningful for the regular representation (PG, G) in action-type geometry.

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